The Detection of Signals in Gaussian Noise

W. L. Root

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In this Chapter we attempt a severely limited but rather careful treatment of what might be now called the classical theory of detection of signals in Gausian noise. It is assumed that the reader has already some acquaintance with the statistical point of view in detection theory and even with the specific problem of detecting a signal in Gaussian noise (at about the level that would be attained from a reading of Chapter 14 of [1]). Thus, the first part of the development will be covered quickly, to serve chiefly as a review and an introduction of notation.

From the point of view of mathematical statistics, problems in detection theory are problems of statistical inference. The kinds of statistical inference problems we choose to be concerned with here are those of hypothesis testing and parameter estimation. For example, the choice of one signal from a finite alphabet of possible signals in a digital data-link transmission is a choice of one hypothesis among several; the measurement of radar range, or of the relative strength of one of various paths in a multipath radio transmission, is the estimation of a parameter. We shall arbitrarily restrict the class of decision rules to certain ones based directly on likelihood ratios, with or without the introduction of loss functions and a priori probabilities. These decision rules can in many instances be justified by the usual statistical criteria, e.g., as being admissible, or as being Bayes solutions for given assignments of loss and a priori probability, but we shall not enter into any discussion of these matters here (for general statistical orientation see, e.g., Chap. 1 of [2] and [3]). For the situations to be considered it is felt the likelihood procedure almost can be justified intuitively.

One kind of mathematics encountered in these problems of statistical inference on Gaussian stochastic processes is standard L_2-space theory (Hilbert space theory), partly because Gaussian random variables are determined by their

*This report will appear as a chapter in a forthcoming book to be edited by A.V. Balakrishnan.

first two moments, and partly because we require that signals have finite energy for finite time periods. It is intended that the development to follow be precise and fairly complete as far as the L_2 parts of the theory are concerned. Another kind of mathematics encountered is more purely measure theoretic, involving, e.g., Radon-Nikodym derivatives and martingales (see [4] for definitions and properties). There will be little discussion of these measure-theoretic aspects, and the reader will not be hampered very much if he lacks knowledge of such things.

The basic equation for each type of problem to be considered here is

$$w(t) = s(t;\alpha) + n(t), \qquad \tau_1 \leq t \leq \tau_2 , \tag{1}$$

where w(t) represents the received waveform, i.e., the raw signal into the receiver or processer; $s(t;\alpha)$ represents the intelligence-bearing signal; n(t) represents the added noise, and α is a parameter or index. If for each value of the parameter or index α, $s(t;\alpha)$ is a known function of t, we call it a sure signal; if for each value of α, $s(t;\alpha)$ is a stochastic process with known (or partially known) statistics, we call it a stochastic signal. We shall consider only sure-signal-in-noise problems in which the added noise is Gaussian, and stochastic-signal-in-noise problems in which both the signal and the added noise are Gaussian--and in fact jointly Gaussian--, with considerably more emphasis on the former.

Likelihood Tests for Sure Signals in Noise

In Eq. (1) $s(t;\alpha)$ is assumed to be a known real-valued continuous function of $t \in [\tau_1, \tau_2]$ for each α in some, as yet unspecified, parameter set A, and n(t) to be a real-valued Gaussian random process with continuous autocorrelation function on $[\tau_1, \tau_2]$, and with mean value identically zero. It is further assumed n(t) is separable and measurable ([4], Chap. 2). We denote the autocorrelation function of n(t) by

$$R(t, s) = En(t)\, n(s) . \tag{2}$$

The integral operator, R, on $L_2[\tau_1, \tau_2]$* which is defined by

$$[Rx](t) = \int_{\tau_1}^{\tau_2} R(t, u)\, x(u)\, du \,, \qquad \tau_1 \leq t \leq \tau_2 \tag{3}$$

for any $x \in L_2[\tau_1, \tau_2]$ is a self-adjoint, positive-definite, Hilbert-Schmidt operator [5]. Thus it has real eigenvalues $\lambda_n \geq 0$ and associated eigenfunctions $\phi_n(t)$ (which can be chosen to be real-valued) satisfying

$$[R\phi_n](t) = \int_{\tau_1}^{\tau_2} R(t, s)\phi_n(s)ds = \lambda_n \phi_n(t), \qquad \tau_1 \leq t \leq \tau_2 \ .$$

Associated with each λ_n there are at most a finite number of linearly independent $\phi_n(t)$; the $\phi_n(t)$ can be chosen to be real and to form an orthonormal set, and the λ_n are either finite in number or have a limit point at zero. We shall assume further that R is strictly positive definite, i.e., that $Rx = 0$ implies $x = 0$ in $L_2[\tau_1, \tau_2]$. Some further comment is made in the Appendix about this point. Then the set of eigenfunctions $\phi_n(t)$ associated with non-zero eigenvalues λ_n can be taken to be a complete orthonormal set. This assumption of strict definiteness holds, for example, if $n(t)$, $\tau_1 \leq t \leq \tau_2$, is a section of a stationary process formed by passing white noise through a realizable, time-invariant filter (i.e., if it is a process of moving averages formed with a kernel which vanishes on a half-line). In any event removal of this condition complicates the following material only superficially.

One way to treat a maximum-likelihood statistical inference problem based on Eq. 1 when $s(t;\alpha)$ is a sure signal is to expand the Gaussian noise process on the interval $[\tau_1, \tau_2]$ in a Fourier series with respect to the $\{\phi_n(t)\}$ (the Karhunen-Loève expansion), which effectively diagonalizes the problem. Then one can easily write probability densities and likelihood ratios for the first N random coefficients in the expansion and pass to the limit. This technique is due to Grenander [6], for a heuristic account of the details in problems

*We use the notation $L_2[a, b]$ to denote the L_2 space with respect to Lebesgue measure on the interval $[a, b]$.

such as we are considering, see [1], Chapter 14. We shall state results here in terms of the Karhumen-Loève expansion, although there are other elegant formulations, such as for example the one in terms of reproducing-kernel Hilbert spaces [7].

Let w_k be the random Fourier coefficient of the received wave-form with respect to $\phi_k(t)$,

$$w_k = \int_{T_1}^{T_2} w(t)\, \phi_k(t)\, dt\ . \tag{4}$$

The w_k exist with probability one for any value of α since (see [4], Theorem 2.7)

$$\int_{T_1}^{T_2} E\,|w(t)\,\phi_k(t)|\,dt \le \int_{T_1}^{T_2} |s(t;\alpha)|\cdot|\phi_k(t)|\,dt + \int_{T_1}^{T_2} E|n(t)|\cdot|\phi_k(t)|\,dt$$

$$\le \left(\int_{T_1}^{T_2} |s(t;\alpha)|^2\,dt\right)^{\frac{1}{2}} + \int_{T_1}^{T_2} En^2(t)\cdot|\phi_k(t)|\,dt$$

$$\le \left(\int_{T_1}^{T_2} |s(t;\alpha)|^2\,dt\right)^{\frac{1}{2}} + \left(\int_{T_1} R^2(t,t)\,dt\right)^{\frac{1}{2}} < \infty .$$

Also the w_k are jointly Gaussian and independent for each value of α ([1], Chap. 14). Let $s_k(\alpha)$ be the k'th Fourier coefficient of the signal with respect to the $\{\phi_k(t)\}$,

$$s_k(\alpha) = \int_{T_1}^{T_2} s(t;\alpha)\phi_k(t)\,dt \tag{5}$$

and define functions of w, which we shall call the <u>test functionals</u> or <u>test statistics</u>, by

$$f(w;\alpha_0) = \sum^{\infty} \frac{w_k s_k(\alpha_0)}{\lambda_k},\quad \alpha_0 \in A\ . \tag{6}$$

If for each $\alpha \in A$

$$\sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_n} < \infty, \tag{7}$$

the series defining $f(w;\alpha_0)$ converges with probability one and also in mean square with respect to the (Gaussian) probability measures determined by each $\alpha \in A$. In fact

$$E_\alpha \left[\frac{w_n s_n(\alpha_0)}{\lambda_n} \right] = \frac{s_n(\alpha) s_n(\alpha_0)}{\lambda_n} \tag{8}$$

$$\mathrm{var}_\alpha \left[\frac{w_n s_n(\alpha_0)}{\lambda_n} \right] = \frac{s_n^2(\alpha_0)}{\lambda_n} \tag{9}$$

so that by condition (7) and the Schwartz inequality the sum of the mean values of the terms on the right side of Eq. (6) converge, as do their variances. This immediately implies convergence in mean square, and by a standard theorem ([4], Theorem 2.3) implies convergence with probability one. Also one has

$$E_\alpha f(w;\alpha_0) = \sum^{\infty} \frac{s_n(\alpha) s_n(\alpha_0)}{\lambda_n} \tag{10}$$

and

$$\mathrm{var}_\alpha f(w;\alpha_0) = \sum^{\infty} \frac{s_n^2(\alpha_0)}{\lambda_n} \quad . \tag{11}$$

It is not surprising, of course, that the variance of $f(w;\alpha_0)$ is not a function of α, since changing α changes only the mean value of $w(t)$. Since the $f(w;\alpha_0)$ are mean-square convergent sums of mutually independent Gaussian variables, they themselves are Gaussian for any $\alpha_0 \in A$, and for any one of the underlying Gaussian probability measures, corresponding to any $\alpha \in A$.

The significance of the test functionals $f(w;\alpha_0)$ lies in their use in maximum-likelihood inference procedures. In particular, if the series in (7) converges for $\alpha = \alpha_0$ and $\alpha = \alpha_1$ the logarithm of the likelihood ratio (i.e., the Radon-Nikodym derivative of the probability measure corresponding to the parameter value α_0 with respect to the measure corresponding to α_1) exists and is given by

$$\log \ell\,(w;\alpha,\alpha_0) = \lim_{N\to\infty} \log \frac{p(w_1,\ldots,w_N;\alpha_0)}{p(w_1,\ldots,w_N;\alpha_1)}$$

$$= f(w;\alpha_1) - f(w;\alpha_0) + C(\alpha_1,\alpha_0) \tag{12}$$

where $p(w_1,\ldots,w_N;\alpha)$ is the joint Gaussian probability density for $(w_1,\ldots,w_N)$ on the hypothesis α, and $C(\alpha_0,\alpha_1)$ depends on $s(t;\alpha_0)$ and $s(t;\alpha_1)$ but not on $w(t)$. Thus, maximum-likelihood hypotheses tests and parameter estimations will involve comparison of the values of $f(w;\alpha)$ for different values of α, or maximization of $f(w;\alpha)$ for $\alpha \in A$.

If we put

$$f_N(w;\alpha_0) = \sum^{N} \frac{w_k\, s_k(\alpha_0)}{\lambda_k}\,, \quad \alpha_0 \in A, \tag{13}$$

and denote the likelihood ratio for the first N observables, $w_1,\ldots,w_N$, by $\ell_N(w;\alpha,\alpha_0)$, then it is an easy calculation (see [1], Chapter 14) to show that

$$\log \ell_N(w,\alpha,\alpha_0) = \log \frac{p(w_1,\ldots,w_N;\alpha_0)}{p(w_1,\ldots,w_N;\alpha_1)}$$

$$= f_N(w;\alpha_1) - f_N(w;\alpha_0) + C_N(\alpha_1,\alpha_0) \tag{14}$$

where $C_N(\alpha_1,\alpha_0)$ depends on $s(t;\alpha_0)$ and $s(t;\alpha_1)$ but not on $w(t)$. The fact that the f_N's converge with probability one we have already shown; the fact that the right side of Eq. (14) converges with probability one to the Radon-Nikodym derivative, to give Eq. (12), involves martingale theory and will not be proved here, see [6]. But it is worth noting that, even without the proof of Eq. (12), we have established that the $f(w;\alpha)$ are limits of test functionals yielding maximum-likelihood tests for any finite number of the observable w_k.

The test statistic or test functional $f(w;\alpha)$ can be thought of, of course, as the output of a linear device with input $w(t) = s(t;\alpha) + n(t)$. One conventional way of defining the output signal-to-noise ratio of a device is as the ratio of the square of the mean of the output to its variance; in the present case, denoting this signal-to-noise ratio by β^2, we have from Eqs. (10) and (11)

$$\beta^2(\alpha) = \frac{[E_\alpha f(w;\alpha)]^2}{\operatorname{var}_\alpha f(w;\alpha)} = \sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_k} \quad . \tag{15}$$

Thus, the condition imposed in Eq. (7) is that the signal-to-noise ratio be finite for each signal $s(t;\alpha)$. The parameter $\beta^2(\alpha)$ together with a threshhold value, to be denoted by η, completely determine the probabilities of error for testing the hypothesis, H_1, that a signal $s(t;\alpha_1)$ is present against the hypothesis, H_0, that no signal is present, i.e., $s(t;\alpha_0) \equiv 0$. In fact, in this situation, $s_k(\alpha_0) = 0$, $k = 1, 2, \ldots$, and the likelihood test reduces to comparing

$$f(w;\alpha_1) = \sum^{\infty} \frac{w_k s_k(\alpha_1)}{\lambda_k}$$

with a threshhold η. For convenience, let us suppress the parameter α_1, since it is the only parameter appearing. Since $f(w)$ is a Gaussian random variable on either hypothesis, and since

$$E_0 f(w) = 0 \,, \qquad E_1 f(w) = \sum^{\infty} \frac{s_k^2}{\lambda_k}$$

$$\operatorname{var}_0 f(w) = \operatorname{var}_1 f(w) = \beta^2 \,,$$

one has for the error probabilities

$$\begin{aligned} e_0 &\triangleq \text{Prob}\{\text{announce } H_1 \,|\, H_0 \text{ is true}\} \\ &= \text{Prob}\{f(w) > \eta \,|\, H_0\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\eta/\beta}^{\infty} e^{-u^2/2}\, du \end{aligned} \tag{16}$$

$$e_1 \triangleq \text{Prob}\{\text{announce } H_0 | H_1 \text{ is true}\}$$
$$= \text{Prob}\{f(w) < \eta | H_1\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta/\beta - \beta} e^{-u^2/2}\, du \quad . \tag{17}$$

Now, for any fixed value of η, e_0, which is the false alarm probability, approaches the value $\frac{1}{2}$ as $\beta^2 \to \infty$, and e_1, which is the probability of miss, approaches the value zero as $\beta^2 \to \infty$. The fact that $e_0 \to \frac{1}{2}$ is an accident of normalization (the variance of f(w) has not been normalized). However, either by appropriately normalizing f(w) or changing the threshhold as β changes, we see that tests can be devised such that both error probabilities go to zero as $\beta^2 \to \infty$. For example, this is accomplished if one takes $\eta = \eta(\beta) = c\beta^{3/2}$, where c is any constant.

The parameter β^2 plays exactly the same role for a detection problem in which the noise has an arbitrary colored spectrum as does the signal-energy-to-noise-power-per-cycle ratio for a detection problem in which the noise is white. By a heuristic argument which is rather obvious, the white noise case can be viewed as an ideal limiting situation in which the λ_k are all equal, and in fact are equal to the noise power per cycle. (To see this, simply substitute $N_0\delta(t)$ for the autocorrelation function in Eq. (3), where N_0 is noise power per cycle.) A rigorous treatment of white noise is not possible within the mathematical structure used here, but it is possible with the use of generalized functions (distributions) and generalized random variables, and leads to the same answers as are obtained purely formally.

Singularity and Non-Singularity of Tests

The condition (7) satisfied for $\alpha \in A$ not only guarantees the existence of the test functional $f(w;\alpha)$, it also guarantees that the test for the signal $s(t;\alpha)$ against the zero signal is non-singular. For the test to be non-singular means that, except for a set of sample functions w(t) of probability zero (on either hypothesis) one cannot tell with probability one after observing w(t) whether the signal $s(t;\alpha)$ was present, or whether there was zero signal. More succinctly, non-singularity means that the test can never be accomplished with zero error probability. The converse is also true, if the condition (7) is not satisfied, $\Sigma \frac{s_k^2(\alpha)}{\lambda_k} = +\infty$, it is possible in principle to tell with probability one after observing w(t) (except for a set of w(t) of probability zero on either hypothesis) whether the signal $s(t;\alpha)$ was or was not present. In the first instance, one says that the two Gaussian measures on the space of sample functions, corresponding to the two different hypotheses, are equivalent; in the second instance, one says they are totally singular*.

We shall not prove the first assertion, affirming non-singularity (see reference in previous footnote) although there is a strong indication of it in the fact that the error probabilities, e_0 and e_1, which were calculated above, are not zero. The converse statement is of some academic interest and it is easy to prove, so a proof of it will be sketched. Basically, what has to be shown to prove total singularity is that one can describe two measurable sets A_0 and A_1 of sample functions w(t) such that

1) $A_0 \cap A_1 = \phi$ (the null set)

and

2) $P_0(A_0) = 1, \; P_1(A_1) = 1$

where P_0 and P_1 denote probabilities with respect to the hypotheses $H_0(s(t,\alpha_0) \equiv 0)$ and $H_1(s(t,\alpha_1))$. Then it will follow immediately that $P_0(A_1) = 0$ and $P_1(A_0) = 0$, since A_0 is contained in the complement of A_1, and vice versa. Hence, if one takes the partition of the sample space given by A_0 and A_0^c (the complement of A_0) as defining the statistical test: choose H_0 if the

*For mathematical definitions of equivalence and singularity of measures see [8]. It is rather generally true that two Gaussian measures on spaces of functions are either equivalent or totally singular--the partially singular case in which a set of observations of probability neither zero nor one yield a sure inference while the complementary set of observations does not, does not occur. The theorem is due to Hajek [9] and Feldman [10]. For a discussion of this matter, see [11] and [12].

observed sample function falls in A_0, choose H_1 if it falls in A_1, one can make a correct decision with probability one. In fact

$$P_0(A_0) = 1, \qquad P_0(A_0^c) = 0$$

$$P_1(A_0^c) = 1, \qquad P_1(A_0) = 0 \, .$$

This partition need not be the only one yielding a singular test.

For convenience, put $s_k(\alpha_1) = s_k$. We have $s_k(\alpha_0) = 0$ for all k. Now set

$$z_N = \frac{f_N(w;\alpha_1)}{\left(\sum^N \frac{s_n^2}{\lambda_n}\right)^{\frac{2}{3}}} = \frac{\sum^N \frac{w_n s_n}{\lambda_n}}{\left(\sum^N \frac{s_n^2}{\lambda_n}\right)^{\frac{2}{3}}} \, .$$

z_N is a Gaussian random variable under either hypothesis, and

$$E_0 z_N = 0, \quad E_1 z_N = \left(\sum^N \frac{s_n^2}{\lambda_n}\right)^{\frac{1}{3}}$$

$$\mathrm{var}_0 z_N = \mathrm{var}_1 z_N = \left(\sum^N \frac{s_n^2}{\lambda_n}\right)^{-\frac{1}{3}}$$

By hypothesis, $\lim\limits_{N\to\infty} \sum^N \frac{s_n^2}{\lambda_n} = +\infty$. Hence $E_1 z_N \to +\infty$, var $z_N \to 0$. An application of the Chebychev inequality will now show that for arbitrary $\epsilon > 0$, and for arbitrarily large $M > 0$, $P_1\{ z_N > M\} > 1 - \epsilon$ and $P_0\{ |z_N| < \epsilon \} > 1 - \epsilon$ for sufficiently large N. That is $z_N \to \infty$ in probability under hypothesis H_1, and $z_N \to 0$ in probability under hypothesis H_0. Then there is a subsequence $\{z_{N'}\}$ of the sequence $\{z_N\}$ which converges to $+\infty$ with probability one (P_1) and to zero with probability one (P_0). Now let A_0 be the set of sample functions w(t) for which $z_{N'} = z_{N'}(w)$ converges to zero, and let A_1 be the set for which $z_{N'}(w)$ converges to $+\infty$. These sets satisfy all the conditions for the sets A_0 and A_1 required above. All that has been done, really, is to formalize the limit situation indicated by the error probability calculations.

The above observations may be extended trivially to cover the case of two arbitrary mean-value functions $s(t;\alpha_0)$, $s(t;\alpha_1)$. The necessary and sufficient condition that the problem of testing between the two hypotheses corresponding to these two signals is nonsingular is

$$\sum^{\infty} \frac{(s_k(\alpha_1) - s_k(\alpha_0))^2}{\lambda_k} < \infty \tag{18}$$

This condition follows immediately because in principle one can consider $w(t) - s(t;\alpha_0)$, which has mean value zero or $s(t;\alpha_1) - s(t;\alpha_0)$ on the hypotheses H_0 and H_1, respectively. The condition (7) for all $\alpha \in A$ implies that (18) is satisfied for all pairs α_0, α_1 belonging to A.

One has the feeling that a properly posed mathematical model should not permit the possibility of perfect signal detection in the presence of noise, unless there are additional constraints that rule out this possibility in the analysis of any physical situation. Arguments that such constraints exist can be given (See, e.g., [11] and [13]); it must be noted however that they are necessarily extra-mathematical, for within the mathematical framework, singularity is possible. It should also be noted that, except for the case of white noise, finiteness of the signal energy is not sufficient to rule out singularity, because the signal energy is given by $\sum^{\infty} s_k^2(\alpha)$, the convergence of which says nothing about the convergence of $\sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_k}$, since the $\lambda_k \to 0$.

The following theorem [15] can be used as a departure point in arguing against the possibility of singularity when the Gaussian noise is stationary, but it also has other uses.

Let n(t) be a Gaussian process satisfying all the conditions previously imposed, plus stationarity. We denote its autocorrelation function by R(t), and require in addition that

i) $\int_{-\infty}^{\infty} |R(t)|\, dt < \infty$

ii) the integral operators R_T defined by

$$(R_T x)(t) = \int_{-T/2}^{T/2} R(t-u)x(u)du, \quad -T/2 \le t \le T/2,$$

for $x(u) \in L_2\,[-T/2,\ T/2]$ have zero null space for any $T > 0$.

By (i), the spectrum of n(t) is absolutely continuous and the spectral density is

$$\Psi(f) = \int_{-\infty}^{\infty} e^{i2\pi ft} R(t)\, dt. \tag{19}$$

Let $\{\phi_n(t;T)\}$ be a complete orthonormal set of (real) eigenfunctions of R_T and $\lambda_n(T)$ be the corresponding eigenvalues, i.e.,

$$\int_{-T/2}^{T/2} R(t-u)\,\phi_n(u;T)dt = \lambda_n(T)\,\phi_n(t;T), \quad -T/2 \le t \le T/2 . \tag{20}$$

Let s(t) be a real-valued function of integrable square, $-\infty < t < \infty$, with Fourier transform (in the sense of Plancherel theory)

$$S(f) = \underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{i2\pi ft} s(t)dt . \tag{21}$$

and define

$$s_n(T) \triangleq \int_{-T/2}^{T/2} s(t)\,\phi_n(t;T)dt . \tag{22}$$

<u>Then,</u> $\sum \frac{s_n^2(T)}{\lambda_n(T)}$ <u>is a non-decreasing function of T which converges monotonically to</u>

$$\int_{-\infty}^{\infty} \frac{|S(f)|^2}{\Psi(f)}\, df$$

if this integral exists, and to $+\infty$ if it does not.

To prove the theorem we introduce a new set of orthonormal functions. Let $H(f) = \sqrt{\Psi(f)}$, $H(f) \geq 0$. Then, since $\Psi(f)$ is absolutely integrable, $H(f)$ is of integrable square, $-\infty < f < \infty$. It has therefore an inverse Fourier transform

$$h(t) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} H(f) e^{-i2\pi ft} \, df.$$

Then,

$$H(f) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} h(t) e^{i2\pi ft} \, dt$$

and

$$R(t) = \int_{-\infty}^{\infty} e^{-i2\pi ft} \Psi(f) \, df$$

$$= \int_{-\infty}^{\infty} h(u) \, h(t-u) \, du \tag{23}$$

(see [14], Theorem 64). $h(t)$ is an even, real-valued function. Now consider the functions

$$\eta_n(t;T) \triangleq \frac{1}{\sqrt{\lambda_n(T)}} \int_{-T/2}^{T/2} h(t-u) \phi_n(u;T) du, \quad -\infty \leq t \leq \infty. \tag{24}$$

Lemma 1. The functions $\eta_n(t;T)$, for any fixed T, are orthonormal on the interval $(-\infty, \infty)$.

In fact,

$$\int_{-\infty}^{\infty} \eta_n(t;T) \overline{\eta_m(t;T)} \, dt$$

$$= \frac{1}{\sqrt{\lambda_n(T)\lambda_m(T)}} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} h(t-u)\phi_n(u;T) du \int_{-T/2}^{T/2} h(t-u')\overline{\phi_n(u';T)} \, du'$$

$$= \frac{1}{\sqrt{\lambda_n(T)\lambda_m(T)}} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \phi_n(u;T)\overline{\phi_m(u';T)} \int_{-\infty}^{\infty} h(t-u)h(t-u')\,dt\,du\,du' \tag{25}$$

by the Fubini-Tonelli theorem, because $h \in L_2(-\infty, \infty)$ and the ϕ_n's $\in L_2[-T/2, T/2]$ and hence $\in L_1[-T/2, T/2]$. From Eq. (23), the right side of Eq. (25) reduces to

$$\frac{1}{\sqrt{\lambda_n(T)\lambda_m(T)}} \int_{-T/2}^{T/2} \phi_n(u;T)\overline{\phi_m(u';T)} R(u-u')\,du\,du' = \delta_{nm}.$$

Lemma 2. Let $\mathcal{L}_T$ be the closed linear manifold of square-integrable functions spanned by the $\eta_n(t;T)$, $n = 1, 2, \ldots$. If $T' < T$, then $\mathcal{L}_{T'} \subset \mathcal{L}_T$.

Let f_T be a square-integrable function orthogonal to $\mathcal{L}_T$. We show that it is orthogonal to $\mathcal{L}_{T'}$. We have

$$\int_{-\infty}^{\infty} \overline{\eta_n(t;T)}\, f(t)\,dt = 0 \quad , \quad n = 1, 2, \ldots \tag{26}$$

Substituting for η_n gives

$$\int_{-\infty}^{\infty} \int_{-T/2}^{T/2} h(t-u)\,\overline{\phi_n(u;T)}\, f(t)\,du\,dt$$

$$= \int_{-T/2}^{T/2} \overline{\phi_n(u;T)} \int_{-\infty}^{\infty} h(t-u)\, f(t)\,dt\,du = 0, \quad n = 1, 2, \ldots .$$

where the change of order of integration is justified because h and f are L_2 functions, and $\phi_n \in L_1[-T/2, T/2]$. The ϕ_n's are complete on $[-T/2, T/2]$, hence

$$\int_{-\infty}^{\infty} h(t-u)\, f(t)\,dt = 0 \tag{27}$$

a.e. on $[-T/2, T/2]$. A fortiori this integral is zero for a.e. t satisfying $|t| \le T'/2 < T/2$, so that

$$\int_{-T'/2}^{T'/2} \overline{\phi_n(t;T')} \int_{-\infty}^{\infty} h(t-u)\, f(t)\,dt\,du = 0, \qquad n = 1, 2, \ldots \quad .$$

Hence, interchanging integrations as above,

$$\int_{-\infty}^{\infty} \overline{\eta_n(t;T')}\, f(t)\,dt = 0 \quad , \quad n = 1, 2, \ldots ,$$

which proves $\mathcal{L}_{T'} \subset \mathcal{L}_T$.

Now let $\mathcal{L}_\infty$ be the closed linear manifold in $L_2(-\infty, \infty)$ spanned by $\bigcup_T \mathcal{L}_T$, and let $\mathcal{H}$ be the class of functions in $L_2(-\infty, \infty)$ whose Fourier transforms vanish where H(f) vanishes, except possibly for a set of measure zero. One can immediately verify that $\mathcal{H}$ is a closed linear subspace of $L_2(-\infty, \infty)$.

Lemma 3. $\mathcal{L}_\infty = \mathcal{H}$. If $x \in \mathcal{H}$ and

$$x_n(T) = \int_{-\infty}^{\infty} x(t)\, \overline{\eta_n(t;T)}\,dt, \qquad n = 1, 2, \ldots, \tag{28}$$

then

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \Big| \sum^{\infty} x_n(T)\, \eta_n(t;T) - x(t) \Big|^2 dt = 0 \tag{29}$$

and

$$\lim_{T\to\infty} \sum^{\infty} |x_n(T)|^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{30}$$

where the convergence is monotone from below.

First we show that $\mathcal{L}_\infty = \mathcal{H}$. The Fourier transform $H_n(f;T)$ of $\eta_n(t;T)$ can be written

$$H_n(f;T) = \frac{1}{\sqrt{\lambda_n(T)}}\, H(f)\, \Phi_n(f;T) \qquad \text{a.e.}$$

where Φ_n is the Fourier transform of ϕ_n. Hence $H_n(f;T)$ vanishes wherever H(f) vanishes. Thus $\mathcal{L}_T \subset \mathcal{H}$ for all T, and hence $\mathcal{L}_\infty \subset \mathcal{H}$. Conversely, suppose $g \in \mathcal{H}$ and is orthogonal to $\mathcal{L}_\infty$. Then g is orthogonal to $\mathcal{L}_T$ for all T and hence satisfies Eq. (27). But if G(f) is the Fourier transform of g(t), Eq. (27) implies that the Fourier transform of H(f)G(f) is zero, and hence that H(f)G(f) = 0 a.e. This implies G(f) vanishes a.e., since $g \in \mathcal{H}$, and hence g is the zero element of L_2. Thus, $\mathcal{H} \subset \mathcal{L}_\infty$.

Now let J_T be the projection operator on the subspace $\mathcal{L}_T$. The family $\{J_T\}$, as $T \to \infty$, is a monotone family of projection operators, which converges strongly to the operator J_∞, the projection on $\mathcal{L}_\infty$, i.e.,

$$\lim_{T\to\infty} J_T x = J_\infty x$$

for every $x \in \mathcal{L}_\infty$. Then, since $\mathcal{L}_\infty = \mathcal{H}$, Eq. (29) holds, and it in turn implies Eq. (30). The convergence is monotone from below since the $\mathcal{L}_T$ are monotone increasing.

An application of Lemma 3 proves the theorem. First we note that if $x \in \mathcal{L}_\infty$ and X(f) is the Fourier transform of x(t), then

$$x_n(T) = \int_{-\infty}^{\infty} x(t)\,\overline{\eta_n(t;T)}\,dt = \int_{-\infty}^{\infty} X(f)\,\overline{H_n(f;T)}\,df$$

and

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Now suppose that $\frac{S(f)}{H(f)} \in L_2$ and is defined to be zero at all values of f for which H(f) vanishes (at such values S(f) must necessarily vanish or the ratio does not belong to L_2). Put $x(f) = \frac{S(f)}{H(f)}$. Then

$$x_n(T) = \int_{-\infty}^{\infty} \frac{S(f)}{H(f)} \cdot \frac{1}{\sqrt{\lambda_n(T)}} \quad H(f)\,\overline{\Phi_n(f;T)}\,df$$

$$= \frac{1}{\sqrt{\lambda_n(T)}} \int_{-T/2}^{T/2} s(t)\phi_n(t;T)\,dt = \frac{s_n(T)}{\sqrt{\lambda_n(T)}} \quad .$$

Then Eq. (30) becomes

$$\lim_{T\to\infty} \sum^{\infty} \frac{s_n^2(T)}{\lambda_n(T)} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\Psi(f)}\, df. \tag{31}$$

Conversely, if

$$\lim_{T\to\infty} \sum^{\infty} \frac{s_n^2(T)}{\lambda_n(T)}$$

exists, one can identify in each $\mathcal{L}_T$ an element whose Fourier coefficients with respect to each $\eta_n(t;T)$ is

$$\frac{s_n(T)}{\sqrt{\lambda_n(T)}} .$$

The resulting sequence of elements, say $\{x_k\}$, corresponding to an arbitrary sequence $\{T_k\}$, $T_k \to \infty$, can be shown to converge to an element of $\mathcal{L}_\infty$ with squared norm given by the right side of Eq. (31). This is equivalent to saying that if the integral on the right side of Eq. (31) diverges, the limit on the left is $+\infty$, which completes the proof of the theorem.

The intuitive content of the theorem is that as the observation interval increases, the signal-to-noise ratio increases (even with a fixed signal that vanishes outside a finite interval) and converges to a signal-to-noise ratio which can be expressed in terms of Fourier transforms of signal and autocorrelation function. This limiting form has long been known to be the signal-to-noise ratio for the detection problem with infinite observation interval [16].

The Matched Filter

Although we have one form for the test functionals, given by Eq. (6), it is important to re-express the $f(w;\alpha)$ in a form more suitable for implementation, particularly for analogue implementation. We now derive the form that permits interpretation as the matched filter, and also, for the case of stationary noise, find an asymptotic expression for it related to the result of the theorem in the last section.

Obviously, $f(w;\alpha)$ can also be written

$$f(w;\alpha) = \lim_{N\to\infty} f_N(w;\alpha) = \lim_{N\to\infty} \int_{\tau_1}^{\tau_2} w(t) \sum^{N} \frac{s_n(\alpha)\phi_n(t)\,dt}{\lambda_n}$$

where $f_N(w;\alpha)$, a "truncated test functional", is defined to be the sum on the right side of Eq. (6) limited to the first N terms. Setting

$$g_N(t;\alpha) = \sum^{N} \frac{s_n(\alpha)\,\phi_n(t)}{\lambda_n} \tag{32}$$

gives

$$f_N(w;\alpha) = \int_{\tau_1}^{\tau_2} w(t)\, g_N(t;\alpha)dt \tag{33}$$

where $g_N(t;\alpha)$ satisfies the equation

$$\int_{\tau_1}^{\tau_2} R(t, u)\, g_N(u;\alpha)du = \sum^{N} s_n(\alpha)\phi_n(t), \quad \tau_1 \le t \le \tau_2 \ . \tag{34}$$

$g_N(t;\alpha)$ thus defined is a function of integrable square on $[\tau_1, \tau_2]$, and it appears in Eq. (33) as a weighting function against which to average $w(t)$ to get $f_N(w;\alpha)$, the approximation to the test functional.

A necessary and sufficient condition that $g_N(t;\alpha)$ converges in $L_2[\tau_1, \tau_2]$ to a function $\tilde{g}(t;\alpha)$ in $L_2[\tau_1, \tau_2]$ is

$$\sum^{\infty} \frac{s_n^2(\alpha)}{\lambda_n^2} < \infty \tag{35}$$

Furthermore, condition (35) is necessary and sufficient that there exist a square-integrable function $g(t;\alpha)$ satisfying

$$f(w;\alpha) = \int_{\tau_1}^{\tau_2} w(t)\, g(t;\alpha)dt \ , \tag{36}$$

and $\tilde{g}(t;\alpha)$ is such a $g(t;\alpha)$. In the case we are considering where R is strictly definite, $g = \tilde{g}$ is unique as an element of $L_2[\tau_1, \tau_2]$. Condition (35) is also necessary and sufficient that the integral equation

$$\int_{\tau_1}^{\tau_2} R(t, u)\, g(u;\alpha)\, du = s(t;\alpha), \qquad \tau_1 \leq t \leq \tau_2 \tag{37}$$

have a solution $g(u;\alpha)$ of integrable square (again, with R strictly definite this solution is unique), and any g satisfying Eq. (37) satisfies Eq. (36) and vice versa. These facts about the limit behavior in $L_2[\tau_1, \tau_2]$, all of which are quite easy to prove, were given by Grenander in his basic paper [6]; the condition for a square-integrable solution to Eq. (37) is a classical theorem of Picard.

Since the $\lambda_n \to 0$, any $s(t;\alpha)$ which satisfies (35) also satisfies the condition for non-singularity (7). Now, putting

$$\begin{aligned} g^*(\tau_2 - t;\alpha) &= g(t;\alpha), \quad \tau_1 \leq t \leq \tau_2 \\ &= 0, \text{ otherwise} \end{aligned}$$

one has from Eq. (36)

$$f(w;\alpha) = \int_{\tau_1}^{\tau_2} g^*(\tau_2 - t;\alpha)\, w(t)\, dt\,, \tag{38}$$

which yields the interpretation that the test statistic $f(w;\alpha)$ is the output of a time-invariant linear filter with impulse response $g^*(t)$, which is turned on at time τ_1 and read at time τ_2. The filter thus described is the well-known matched filter for the signal $s(t;\alpha)$ with added noise $n(t)$. In building a device to detect coherently sure signals in noise it is impossible of course to implement the infinite series of Eq. (6) as such, whereas it is possible in principal to implement the matched filter if it exists. It is a nearly trivial comment that if any linear filter is used as a detection device in the role of a matched filter, as specified in Eq. (38), it is the correct matched filter for the given noise and some $s(t) \in L_2[\tau_1, \tau_2]$; and this s(t) satisfies (35) and (37).

In case the noise is stationary there is for large T an asymptotic form of the matched filter (or simply, an asymptotic matched filter) whose characteristics can be described entirely in terms of the signal and noise spectra--thus obviating the necessity of solving the integral equation for $g(t;\alpha)$ if T is large enough. This statement is made precise below in a theorem. Heuristically, one can argue from Eq. (37) that as $T \to \infty$, the Fourier transform G(f) of g(t) ought to look like the ratio of the transforms of s(t) and R(t). This is in fact true, as follows. We do not assume the conditions (7) and (35), though (7) is implied by the hypotheses we do make.

Let $s(t;\alpha)$ be a real-valued function for each $\alpha \in A$ which vanishes identically for $|t| > T_0$, T_0 a fixed positive number, and is square-integrable. Let n(t), $-\infty < t < \infty$, be a real-valued stationary Gaussian process with mean zero. Let R(t), the autocorrelation function of n(t), have the properties that

i) $\int_{-\infty}^{\infty} |R(t)| dt < \infty$

ii) the integral operator, R, with kernel $R(t - u)$, on $L_2[-\frac{T}{2}, \frac{T}{2}]$ has zero null space for any $T > 0$.

Suppose further that

iii) $\int_{-\infty}^{\infty} \frac{|S(f;\alpha)|^2}{\Psi^2(f)} df < \infty, \alpha \in A$

and let $\tilde{g}(t;\alpha)$ be the inverse Fourier transform of $\frac{S(f;\alpha)}{\Psi(f)}$ in the sense of the Plancherel theorem. Then for sufficiently large T the linear functional

$$\tilde{f}_T(w;\alpha) = \int_{-T/2}^{T/2} w(t) \tilde{g}(t;\alpha) dt \tag{39}$$

has a (Gaussian) distribution which is arbitrarily close, in the sense of the Levy metric, to the distribution of $f_T(w;\alpha)$ for any actual signal $s(t;\alpha_0)$. Here we have indicated explicitly the dependence of the test functional f on the observation interval length. Further, if $\alpha_1, \ldots, \alpha_N$ all belong to A, for sufficiently large T the test functionals $\tilde{f}_T(w;\alpha_i)$, $i = 1, \ldots, N$, have a joint Gaussian distribution which is arbitrarily close to that of $f_T(w;\alpha_i)$, $i = 1, \ldots, N$, for any actual signal $s(t;\alpha_i)$, $i = 1, \ldots, N$, if the latter joint distribution is non-singular.

The first assertion is a special case of the second. The functionals $\tilde{f}_T(w;\alpha)$ are jointly Gaussian as are the functionals $f_T(w;\alpha)$, so it is sufficient to show (in the presence of the non-singularity hypothesis) that the means, variances and covariances of the $\tilde{f}_T(w;\alpha_i)$ approach those of the $f_T(w;\alpha_i)$.

We note again that i) implies that the spectrum of the noise process is absolutely continuous and that the spectral density $\Psi(f)$ is the Fourier transform of R. Also, from iii)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{|S(f;\alpha)|^2}{\Psi(f)}\, df &= \int_{\Psi(f)>1} + \int_{\Psi(f)\le 1} \\ &\le \int_{-\infty}^{\infty} |S(f;\alpha)|^2\, df + \int_{-\infty}^{\infty} \frac{|S(f;\alpha)|^2}{\Psi^2(f)} \\ &< \infty\end{aligned}$$

so the non-singularity condition is satisfied for each α.

In the remainder of the proof we again indicate the dependence of the eigenvalues and Fourier coefficients on T by writing $\lambda_n(T)$, $s_n(T;\alpha)$, $w_n(T)$. By the theorem in the preceding section i) and ii) plus the conditions on $s(t;\alpha)$ guarantee that

$$\sum^{\infty} \frac{s_n^2(T;\alpha)}{\lambda_n(T)} \uparrow \int_{-\infty}^{\infty} \frac{|S(f;\alpha)|^2}{\Psi(f)}\, df \qquad \text{as } T \to \infty .$$

This theorem is trivially generalized to give

$$\sum^{\infty} \frac{s_n(T;\alpha)s_n(T;\alpha_0)}{\lambda_n} \rightarrow \int_{-\infty}^{\infty} \frac{S(f;\alpha)\ \overline{S(f;\alpha_0)}}{\Psi(f)}\, df$$

by writing

$$S(f;\alpha)\ \overline{S(f;\alpha_0)} = \tfrac{1}{4}\big[\,|S(f;\alpha) + S(f;\alpha_0)|^2 - |S(f;\alpha) - S(f;\alpha_0|^2$$
$$+ i\,|S(f;\alpha) + iS(f;\alpha_0)|^2 - i\,|S(f;\alpha) - iS(f;\alpha_0|^2\big]$$

and the corresponding decomposition for $s_n(T;\alpha)\ s_n(T;\alpha_0)$, and observing that the individual terms thus formed satisfy the stated theorem. We then have from Eqs. (10) and (11) that

$$E_\alpha f_T(w;\alpha_0) \rightarrow \int_{-\infty}^{\infty} \frac{S(f;\alpha)\ \overline{S(f;\alpha_0)}}{\Psi(f)} \qquad \text{as } T \rightarrow \infty$$

and

$$\text{covar}\ \ f_T(w;\alpha_0)\, f_T(w;\alpha_1) \rightarrow \int_{-\infty}^{\infty} \frac{S(f;\alpha_0)\ \overline{S(f;\alpha_1)}}{\Psi(f)}\, df \qquad \text{as } T \rightarrow \infty$$

Since the restriction of $\tilde{g}(t;\alpha)$ belongs to $L_2[-\frac{T}{2}, \frac{T}{2}]$ it belongs to $L_1[-\frac{T}{2}, \frac{T}{2}]$, and this fact plus the fact that $E_\alpha|n(t)|$ is a finite constant guarantees the existence of $\tilde{f}_T(w;\alpha_0)$ with probability one, and justifies averaging inside the integral. One has

$$E_\alpha \tilde{f}_T(w;\alpha_0) = \int_{-T/2}^{T/2} Ew(t)\ \tilde{g}(t;\alpha_0)\ dt$$
$$= \int_{-\infty}^{\infty} s(t;\alpha)\ \tilde{g}(t;\alpha_0)\ dt$$
$$= \int_{-\infty}^{\infty} \frac{S(f;\alpha)\ \overline{S(f;\alpha_0)}}{\Psi(f)}\, df$$

for $T > 2T_0$. Also,

$$\operatorname{covar} \tilde{f}_T(w;\alpha_0)\tilde{f}_T(w;\alpha_1) = \int_{-T/2}^{T/2}\int R(t-u)\,\tilde{g}(t;\alpha_0)\,\overline{\tilde{g}(u;\alpha_1)}\,dt\,du$$

$$= \int_{-\infty}^{\infty}\int R(t-u)\,\tilde{g}_T(t;\alpha_0)\,\overline{\tilde{g}_T(u;\alpha_1)}\,dt\,du$$

where $\tilde{g}_T(t;\alpha) = \tilde{g}(t;\alpha), \quad |t| \le \frac{T}{2}$

$= 0$ otherwise

We let $< \cdot >$ denote inner product and $\|\cdot\|$ denote norm on $L_2(-\infty,\infty)$, and $*$ denote convolution, so that var $\tilde{f}_T(w;\alpha) = < R*\tilde{g}_T, \tilde{g}_T >$, where we have dropped the α. Since $\|g_T - g\| \to 0$, and the L_1 norm of R, $\|R\|_1$, is finite we have

$$|< R*\tilde{g}, \tilde{g} > - < R*\tilde{g}_T, \tilde{g}_T > |$$

$$\le \|R\|_1 \cdot \|\tilde{g}\| \cdot \|g - g_T\| + \|R\|_1 \cdot \|g - g_T\| \cdot \|g_T\| \to 0$$

as $T \to \infty$.

Here we have used the fact that

$$\|R*g\| \le \|R\|_1 \cdot \|g\| .$$

Now $$< R*\tilde{g}, \tilde{g} > = \int_{-\infty}^{\infty} \left(\Psi(f) \cdot \frac{S(f;\alpha_0)}{\Psi(f)}\right) \cdot \frac{\overline{S(f;\alpha_0)}}{\Psi(f)}\,df$$

$$= \int_{-\infty}^{\infty} \frac{|S(f;\alpha_0)|^2}{\Psi(f)}$$

by the integrability conditions on $R(t)$ and $\tilde{g}(t;\alpha_0)$. Thus $f_T(w;\alpha_0)$ and $\tilde{f}_T(w;\alpha_0)$, both of which are Gaussian for all $T > 0$, have means and variances which converge to respectively the same finite limits as $T \to \infty$. A trivial modification of the above argument shows that covar $\tilde{f}_T(w;\alpha_0)\,\tilde{f}_T(w;\alpha_1)$ and covar $f_T(w;\alpha_0)\, f_T(w;\alpha_1)$ converge to the same finite limits also.

Remark If in addition to the hypothesis iii), the function $\frac{S(f;\alpha)}{\Psi(f)}$ is an integral function of exponential type with index $\leq \pi T_1$, then for $T \geq T_1$, the integral equation (37) has a solution in $L_2[-\frac{T}{2}, \frac{T}{2}]$ by a well-known theorem of Paley and Wiener. This condition, which is satisfied, for example, if in addition to iii) and the vanishing of s outside [-T, T] is the reciprocal of a polynomial, is stronger than necessary. In general as far as we know one cannot infer from the hypothesis of this theorem that Eq. (37) does have a solution, that is, that there is properly speaking a matched filter. The existence of the matched filter in the L_2 sense is of course irrelevant in both the statement and proof of the above theorem.

Stability

It is worth finding out whether the performance of a detector is critically dependent upon the noise statistics and signal waveforms being exactly what they are assumed to be. Even though a detector is optimum for a certain signal waveform in Gaussian noise with a certain autocorrelation function, if the error probabilities rise sharply with slight changes in signal waveform or noise structure, the detector will not be in practice very good, because the actual signal and the actual noise are unlikely to be exactly what was assumed. We shall refer to the property of a detector to maintain its performance under shifting conditions as its stability.* We now investigate a little the stability of likelihood detectors using the test functionals $f(w;\alpha)$. To do this properly we should allow for more or less arbitrary perturbations in the signal and in the noise statistics, but it is difficult to account quantitatively for changes in the distribution of the noise statistics from the Gaussian, so we restrict ourselves to perturbations in the noise statistics which are reflected only by changes in the autocorrelation function, while the noise process remains Gaussian (see

*In the statistical literature, what we are here calling stability is called the robustness of a statistical decision procedure.

[17]).

Suppose first that the condition (35) is satisfied, that the noise autocorrelation function is accurately known, but that there are changes in the signal waveform. It is easy to see that small changes in the signal $s(t;\alpha)$ in the mean square sense are reflected as small changes in the mean of the distribution of $f(w;\alpha)$, which distribution is otherwise unchanged. In fact, let $s(t;\alpha)$, $\alpha \in A$, be the family of nominal signals, with Fourier coefficients $s_k(\alpha)$ with respect to the $\phi_k(t)$. Let the actual signals be $s'(t;\alpha)$ with Fourier coefficients $s'_k(\alpha)$. Then the received waveform is

$$w'(t) = s'(t;\alpha) + n(t) \tag{40}$$

and

$$f(w';\alpha_0) = \sum \frac{(s'_k(\alpha) + n_k)\, s_k(\alpha_0)}{\lambda_k} \quad . \tag{41}$$

$f(w';\alpha_0)$ is gaussian for any $\alpha \in A$; its variance, as before, is $\sum \frac{s_k^2(\alpha_0)}{\lambda_k}$, but its mean is

$$E_\alpha f(w';\alpha_0) = \sum \frac{s'_k(\alpha)\, s_k(\alpha_0)}{\lambda_k} \quad . \tag{42}$$

Then the difference between the nominal mean value of $f(w;\alpha_0)$ and its actual mean is given by

$$\left|E_\alpha f(w';\alpha_0) - E_\alpha f(w;\alpha_0)\right|^2 = \left|\sum \frac{(s'_k(\alpha) - s_k(\alpha))\, s_k(\alpha_0)}{\lambda_k}\right|^2$$

$$\leq \sum \frac{s_k^2(\alpha_0)}{\lambda_k^2} \int_{\tau_1}^{\tau_2} \left|s'(t;\alpha) - s(t;\alpha)\right|^2 dt \tag{43}$$

by the Schwarz inequality. The inequality (43) may also be written in terms of the weighting function $g(t;\alpha)$, since the condition for the existence of a matched filter is satisfied, as

$$\left|E_\alpha f(w';\alpha_0) - E_\alpha f(w;\alpha_0)\right| \leq \|g(\alpha_0)\| \cdot \|s'(\alpha) - s(\alpha)\| \tag{44}$$

where $\|\cdot\|$ denotes the L_2 norm on $[\tau_1, \tau_2]$.

Thus, if $\|s'(\alpha) - s(\alpha)\|$ is small, the mean of the distribution of $f(w;\alpha_0)$ is shifted only slightly, and small changes in the error probabilities result. For the case of simple detection, where the H_0 hypothesis is the no-signal hypothesis and the H_1 hypothesis is that the signal is $s(t;\alpha)$, the error probability e_0 defined in Eq. (16) is unchanged, and the error probability e_1 defined in Eq. (17) is increased by the amount

$$\Delta e_1 = \frac{1}{\sqrt{2\pi}} \int_{\frac{\eta}{\beta}}^{\frac{\eta}{\beta} - \beta + \frac{1}{\beta}(E_\alpha f(w;\alpha) - E_\alpha f(w';\alpha))} e^{-\frac{1}{2}u^2} du \quad , \tag{45}$$

as is readily verified. Δe_1 may be negative, of course.

It is interesting to note that if the condition (35) is not satisfied the detector is completely unstable, in the sense that an arbitrarily small change in $s(t;\alpha)$ in the mean-square sense can (not necessarily will) result in an arbitrarily large change in the mean value of $f(w;\alpha)$. To see this we note first a theorem of Landau ([18] p. 1) to the effect that if $\{p_k\}$ is a given sequence of real numbers such that $\Sigma\, p_k^2 = +\infty$, then there exists a sequence of real numbers $\{q_k\}$ such that $\Sigma\, q_k^2 < +\infty$, but $\Sigma\, p_k q_k$ diverges. To apply this here, suppose $\sum \frac{s_k^2(\alpha)}{\lambda_k^2} = +\infty$, choose $\epsilon > 0$ arbitrarily small and $B > 0$ arbitrarily large. By the Landau theorem there is a sequence $\{q_k\}$ such that

$$\sum q_k^2 = \epsilon^2$$

and

$$\sum \frac{s_k(\alpha) q_k}{\lambda_k} = +\infty.$$

Then put

$$s'_k = s_k(\alpha) + q_k, \quad k = 1, \ldots, N$$

$$= s_k(\alpha), \quad k = N+1, \; N+2, \ldots$$

where N is chosen large enough that

$$\left| \sum^{N} \frac{s_k(\alpha) q_k}{\lambda_k} \right| \geq B.$$

Then $\|s'(\alpha) - s(\alpha)\| = \sum^{N} q_k^2 \leq \sum^{\infty} q_k^2 = \epsilon^2$ while

$$|E_\alpha f(w';\alpha) - E_\alpha f(w;\alpha)| = \left| \sum^{N} \frac{s_k(\alpha) q_k}{\lambda_k} \right| \geq B.$$

Note that the condition for non-singularity need not be violated by either the nominal signal $s(t;\alpha)$ or the actual signal $s'(t;\alpha)$. In fact $\sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_k}$ can converge even though $\sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_k^2}$ diverges, because $\lambda_k \to 0$. Also,

$$\sum^{\infty} \frac{s'^2_k(\alpha)}{\lambda_k} = \sum^{\infty} \frac{s_k^2(\alpha)}{\lambda_k} + 2\sum^{N} \frac{s_k(\alpha) q_k}{\lambda_k} + \sum^{N} \frac{q_k^2}{\lambda_k}$$

will certainly converge if the first term on the right converges, that is if $s(t;\alpha)$ satisfies the non-singularity condition. *

Now let us require again that (35) be satisfied and let us suppose that the signal is exactly as specified, but the noise, though Gaussian with mean zero, has autocorrelation function

*I am indebted to Professor T. Kailath for pointing out to me that the stability discussion does not make much sense unless <u>both</u> the nominal and actual signals (or statistics) are required to satisfy the non-singularity condition. This point is overlooked in [17], where there is some ambiguity in the discussion of the unstable case for Gaussian noise.

$$R'(t, u) = R(t, u) + A(t, u) \tag{46}$$

where $R(t, u)$ is the nominal autocorrelation function, as used in defining $f(w;\alpha)$. It is reasonable to require that $R'(t, u)$ as well as $R(t, u)$ be bounded, hence the $L_2[\tau_1, \tau_2] \times [\tau_1, \tau_2]$ norm of A, to be denoted by $\|A\|$, is necessarily finite,

$$\|A\|^2 = \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} |A(t, u)|^2 \, dt\, du < \infty.$$

In general we let primed quantities refer to actual statistics, unprimed quantities refer to nominal statistics. The test functionals

$$f(w;\alpha) = \sum \frac{s_k(\alpha) w_k}{\lambda_k}$$

are unchanged; λ_k, $\phi_k(t)$ and the Fourier coefficients refer to the nominal autocorrelation $R(t, s)$. For convenience we again consider the case of simple detection; the hypothesis H_0 is the no-signal hypothesis, the hypothesis H_1 is that a signal $s(t)$ is present (we drop the unneeded parameter α). Then, putting

$$f(w) = \sum \frac{s_k w_k}{\lambda_k} \quad , \tag{47}$$

$$E_0' f(w) = E_0 f(w) = 0$$

$$E_1' f(w) = E_1 f(w) = \sum \frac{s_k^2}{\lambda_k}$$

Now, however, letting $n_k = \int_{\tau_1}^{\tau_2} n(t)\, \phi_k(t)\, dt$,

$$\begin{aligned} E' n_k n_j &= E' \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} n(t)\, n(u)\, \phi_k(t)\, \phi_j(u)\, dt\, du \\ &= \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} [R(t, u) + A(t, u)]\, \phi_k(t)\, \phi_k(u)\, dt\, du \\ &= \lambda_k \delta_{kj} + (A\phi_k, \phi_j) \end{aligned}$$

where $(\cdot\,,\cdot)$ is the inner product for $L_2\,[\tau_1, \tau_2]$ and A is the integral operator with kernel A(t, u). Thus,

$$\mathrm{var}'f(w) = E_0' \sum \frac{s_k w_k}{\lambda_k} \sum \frac{s_j w_j}{\lambda_j}$$

$$= \sum_{k,j} \frac{s_k s_j E n_k n_j}{\lambda_k \lambda_j}$$

$$= \sum \frac{s_k^2}{\lambda_k} + \sum_{k,j} \frac{s_k s_j}{\lambda_k \lambda_j} (A\phi_k, \phi_j) \; . \tag{48}$$

Put

$$\Delta = \sum_{k,j} \frac{s_k s_j}{\lambda_k \lambda_j} (A\phi_k, \phi_j). \tag{49}$$

Then

$$|\Delta|^2 \leq \sum_{k,j} \frac{s_k^2}{\lambda_k^2} \frac{s_j^2}{\lambda_j^2} \cdot \sum_{k,j} |A\phi_k, \phi_j)|^2$$

or

$$|\Delta| \leq \sum \frac{s_k^2}{\lambda_k^2} \cdot [\![A]\!] \; . \tag{50}$$

Thus the distribution of f(w) is unchanged except for its variance, under either hypothesis, and the change in the variance is bounded by the right side of (50). Actually, a sharper estimate is possible, for $[\![A]\!]$ may be replaced by the operator norm of A, as is easily seen.

Again we can show that if $\sum \frac{s_k^2}{\lambda_k^2} = +\infty$, complete instability may exist, this time in the sense that an arbitrarily small change in R(t, u) in the mean-square sense can result in an arbitrarily large change in the variance of f(w). Let $\epsilon > 0$ be chosen arbitrarily small and $B > 0$ be chosen arbitrarily large. Suppose a sequence of real numbers $\{c_k\}$ is chosen so that

$$\sum^{\infty} c_k^2 = \epsilon$$

and

$$\sum \frac{c_k s_k}{\lambda_k} = +\infty.$$

Choose N such that $\left|\sum^{N} \frac{c_k s_k}{\lambda_k}\right| > B$, and define an operator A by

$$\begin{aligned}(A\phi_k, \phi_j) &= c_k c_j\ , \quad k, j = 1, \ldots, N \\ &= 0\ , \quad k \text{ or } j = N+1,\ N+2, \ldots\end{aligned}$$

Then

$$\sum_{k,j}^{\infty} |(A\phi_k, \phi_j)|^2 = \sum_{k,j}^{N} c_k^2 c_j^2 \leq \epsilon^2$$

so A is a Hilbert-Schmidt operator (see [19]). It is symmetric, since for $x, y \in L_2[\tau_1, \tau_2]$

$$\begin{aligned}(Ax, y) &= (A \sum x_k \phi_k, \sum y_j \phi_j) \\ &= \sum_{k,j}^{N} x_k \bar{y}_j (A\phi_k, \phi_j) = \sum_{k,j}^{N} x_k \bar{y}_j c_k c_j \\ &= (x, Ay)\end{aligned} \tag{51}$$

where the x_k and y_k are Fourier coefficients of x and y with respect to ϕ_k. A is also positive definite, as is shown by putting y = x in Eq. (51), whence

$$(Ax, x) = \left|\sum x_k c_k\right|^2 \geq 0.$$

Thus A is a symmetric, positive definite integral operator on $L_2[\tau_1, \tau_2]$ with kernel

$$A(t, u) = \sum_{k,j}^{N} c_k c_j \phi_k(t) \phi_j(u)\ . \tag{52}$$

A(t, u) is furthermore continuous, since the $\phi_k(t)$ are continuous. We have then that

$$R'(t, u) = R(t, u) + A(t, u)$$

is a continuous autocorrelation function such that

$$\| R' - R \| \leq \epsilon$$

but such that

$$|\Delta| = \left| \sum^{N} \frac{s_k c_k}{\lambda_k} \right|^2 \geq B^2.$$

The spectral decomposition of the operator R' is different from that of R, but only in the first N eigenvalues and eigenfunctions. Hence, if

$$\sum \frac{s_k^2}{\lambda_k} < \infty$$

then the corresponding series based on the eigenvalues and eigenfunctions of R' is also finite, because the tails of the two series are the same. Thus it is possible in principle to have an unstable detector even though the condition for non-singularity is satisfied both for the nominal and actual noise autocorrelation.

It must be admitted that the exact condition (35) separating stable and unstable operation is in a sense artificial, because it depends on the choice of mean-square difference as a measure of the perturbations of signal and noise autocorrelation (see [17]). It is, of course, exactly the same condition as that permitting the integral form for the test functional (the matched filter). The fact that the two conditions agree is curious and interesting but it depends on treating the whole problem consistently in the context of L_2 spaces.

The M-symbol Likelihood Detector

The multiple hypothesis test problem of choosing one of M possible signals in Gaussian noise is rather special from a statistical point of view, but surprisingly general from a communications and measurement theory point of view. This is because in practical problems the dependence of $s(t;\alpha)$ on α is usually continuous when α is a continuous scalar or vector parameter, and it is often not only permissible but desirable to make a discrete approximation to it, that is to quantize the parameters. A familiar example is the measurement of doppler shift by a radar by using a parallel bank of band-pass filters.

So let us now consider this multiple hypothesis test problem from the point of view already advanced. To start with, consider a test based only on the observables $w_1, \ldots, w_N$. We have $\alpha = 1, 2, \ldots, M$ with each $s(t;\alpha)$ a known signal, and let us adjoin $\alpha = 0$ as the zero-signal hypothesis. Suppose a priori probabilities are assigned so that $\alpha = k$ has probability π_k, $k = 1, \ldots, M$. A likelihood test to determine α is as follows: compute $\pi_\alpha p(w_1, \ldots, w_N;\alpha)$ for $\alpha = 1, \ldots, M$ and choose that hypothesis $\alpha = k$ for which the resulting value is largest. This test has the property that it yields the maximum probability of making a correct decision (among tests using only $w_1, \ldots, w_N$), as may be readily verified. Now the test is not changed if, instead of choosing the largest from among $\pi_\alpha p(w_1, \ldots, w_N;\alpha)$, $\alpha = 1, \ldots, M$, one chooses the largest from among

$$\log \pi_\alpha \frac{p(w_1, \ldots, w_N;\alpha)}{p(w_1, \ldots, w_N;0)} = \sum^N \frac{w_k s_k(\alpha)}{\lambda_k} - \frac{1}{2}\sum^N \frac{s_k^2(\alpha)}{\lambda_k} + \log \pi_\alpha .$$

It is still true in the limit as $N \to \infty$ that this test yields the maximum probability of making a correct decision if condition (7) is satisfied [20].

The detector prescribed by this test consists then of M devices in parallel which compute $f(w;\alpha) - c_\alpha$, $\alpha = 1, \ldots, M$, where each c_α is a predetermined constant, and a comparator to choose the largest (in the arithmetic sense, not in absolute value). Let us assume

$$\sum \frac{s_k^2(\alpha)}{\lambda_k^2} < \infty \quad , \ \alpha = 1, \ldots, M,$$

which implies the condition (7). The detector is then non-singular and stable, and can consist essentially of a bank of M matched filters. If the noise is stationary, the asymptotic forms for the matched filters are applicable. The M outputs are jointly Gaussian, so error probabilities may be calculated directly. For example, the conditional probability of making a correct decision given that the true value of α is 1 is

$$P_1\{f(w;1) - c_1 > f(w;2) - c_2, \ldots, f(w;1) - c_1 > f(w;M) - c_M\}.$$

Put
$$\xi_k = f(w;1) - f(w;k) - c_1 + c_k.$$

Then
$$m_k \triangleq E_1 \xi_k = \int_{\tau_1}^{\tau_2} s(t;1)[g(t;1) - g(t;k)]\,dt - c_1 + c_k \tag{53}$$

and
$$\begin{aligned}\sigma^2_{kj} &\triangleq E_1(\xi_k - m_k)(\xi_j - m_j)\\ &= \int_{\tau_1}^{\tau_2} s(t;1)\,g(t;1)\,dt\\ &\quad - \int_{\tau_1}^{\tau_2} s(t;k)\,g(t;1)\,dt - \int_{\tau_1}^{\tau_2} s(t;j)\,g(t;1)\,dt\\ &\quad + \int_{\tau_1}^{\tau_2} s(t;k)\,g(t;j)\,dt \quad .\end{aligned} \tag{54}$$

The probability of a correct decision is

$$\begin{aligned}P_1\{\xi_2 > 0, \xi_3 > 0, \ldots, \xi_M > 0\} &= P_1\{\xi_2 - m_2 > -m_2, \ldots, \xi_M - m_M > -m_M\}\\ &= \frac{1}{(2\pi)^{\frac{M-1}{2}}\,|\sigma^2|^{\frac{1}{2}}} \int_{-m_M}^{\infty} \cdots \int_{-m_2}^{\infty} \exp[-\tfrac{1}{2}\sum_{2}^{M} \gamma_{ij} x_i x_j]\,dx_2 \cdots dx_M\end{aligned} \tag{55}$$

where the matrix $\Gamma = [\gamma_{ij}]$ is the inverse to the covariance matrix $[\sigma^2_{ij}]$ and $|\sigma^2|$ is the determinant of the covariance matrix.

If one has all the $\sum \frac{s_k^2(\alpha)}{\lambda_k}$ equal, as is often approximately true, and the π_α all equal, then an optimum test results with the c_α all equal, and they may be cancelled out. The detector then has the property of being optimum for arbitrary signal-to-noise ratio, which otherwise would have to enter as a parameter, either with or without an a priori distribution (see [1], Chap. 14).

Stochastic Signals

We suppose now that the signal term in Eq. (1) is random and Gaussian. In particular let

$$w(t) = s(t;\alpha) + n(t), \ \alpha = 1, 2, \ldots, M; \ \tau_1 \le t \le \tau_2 \tag{56}$$

where s(t) is for each α (i.e., each possible signal) a sample function from a real-valued Gaussian process, each with continuous autocorrelation function, and each independent of n(t). Let

$$a(t;\alpha) \triangleq E_\alpha s(t;\alpha) \tag{57}$$

and

$$z(t;\alpha) \triangleq a(t;\alpha) - s(t;\alpha), \ \alpha = 1, 2, \ldots, M\ . \tag{58}$$

That is, $a(t;\alpha)$ is the sure-signal component of the total received signal, and $z(t;\alpha)$ is the purely random part. Denote the correlation functions of the $z(t;\alpha)$ by

$$\Gamma^{(\alpha)}(t, u) \triangleq E_\alpha z(t;\alpha)\, z(u;\alpha) \tag{59}$$

and of the noise by

$$R(t, u) \triangleq En(t)\, n(u) \tag{60}$$

To start with, consider a finite set of observation times, $\{t_k\}$, $t_1 \le t_1 < t_2 < \cdots < t_k \le \tau_2$, from the observation interval $[\tau_1, \tau_2]$ and let $n, z^{(\alpha)}, a^{(\alpha)}, w$ be the column vectors with entries $n(t_k)$, $z(t_k;\alpha)$, $y(t_k;\alpha)$, $w(t_k)$ respectively; e.g.,

$$z^{(\alpha)} = \begin{bmatrix} z(t_1;\alpha) \\ \vdots \\ z(t_K;\alpha) \end{bmatrix}$$

The covariance matrices of the various random vectors, all of which we assume to be non-singular, are denoted by the same letter as the corresponding correlation function, e.g.,

$$\Gamma^{(\alpha)} \triangleq [\Gamma^{(\alpha)}(t_j, t_k)] \triangleq E_\alpha z^{(\alpha)} z^{(\alpha)T} ,$$

where we use the notation A^T for the transpose of the matrix A. Then the probability density function for the observed vector w on the hypothesis α is the K-variate Gaussian density function

$$p(w|\alpha) = \frac{1}{(2\pi)^{\frac{K}{2}} |\Gamma^{(\alpha)} + R|^{\frac{1}{2}}} \exp\left[-\tfrac{1}{2}(w - a^{(\alpha)})^T \cdot (\Gamma^{(\alpha)} + R)^{-1}(w - a^{(\alpha)})\right] \tag{61}$$

where the symbol $|A|$ has been used to denote the determinant of the matrix A. A likelihood test for determining which signal α, $\alpha = 1, \ldots, M$, was sent is then to compare the values taken on by $p(w|\alpha)$ for the waveform actually received, and choose that α which gives the largest value. The $p(w|\alpha)$ may each be given pre-assigned weightings if desired; this amounts of course to comparing the *a posteriori* probabilities of the transmitted signals indexed by α given w, where the weightings are proportional to the *a priori* probabilities of these symbols. In any event for the usual decision procedures, the essential part of the data processing is to determine the $p(w|\alpha)$, or quantities related to them which have the same ordering as the $p(w|\alpha)$ for each w. It is convenient to take the logarithms of the $p(w|\alpha)$; then the determinants enter only as additive constants and the active part of the data processer consists of M channels, each of which computes one of the M quadratic forms,

$$(w - a^{(\alpha)})^T(\Gamma^{(\alpha)} + R)^{-1}(w - a^{(\alpha)}) . \tag{62}$$

In a sense, then, the problem of constructing a good receiver is solved in this simple fashion. Indeed, in case the problem is non-singular a sufficiently fine net of sample points t_n yields an approximation to the "best" answer in the sense that it guarantees the ratio of any two of the $p(w|\alpha)$ will be arbitrarily close, with probability one, to the Radon-Nikodym derivative of the corresponding probability measures. The situation is not entirely satisfactory, however, because the limit theory is complicated. It is difficult to "diagonalize" the problem as was done for sure signals in Gaussian noise. We shall only make reference to some of the literature ([7], [9], [10], [11], [12], [21], [22], [23], [24], [25], [26], [27]) and not discuss the limit theory here for the general case.

There is one very useful idealized special case in which most of the mathematical difficulties do not appear, however; it is the problem of detecting a stochastic signal in white noise. Here one can represent the stochastic signal by its Karhunen-Loeve expansion and, formally at least, also represent the noise as a random Fourier expansion with uncorrelated coefficients in terms of the same orthogonal functions. If the noise is not white over the infinite band this is an approximation. This situation is, in fact, a formal particular case of that in which the covariance functions of the signal and noise determine integral operators as in Eq. (3) which commute. We now look at the detection of stochastic signals in noise under this special assumption of commutativity. In Eq. (56) let $\alpha = 0$ or 1, and let $s(t;o)$ be identically zero and $s(t;1)$ be a Gaussian process with mean zero and continuous autocorrelation function

$$\Gamma(t,u) = Es(t;1)\, s(u;1) \quad . \tag{63}$$

Let $n(t)$ be Gaussian noise with mean zero and continuous autocorrelation function

$$R(t,u) = En(t)\, n(t) \quad .$$

We suppose, in accordance with the above assumption, that the operators R and Γ defined as in Eq. (3) with the autocorrelation functions as kernels satisfy $R\Gamma = \Gamma R$, and that each is strictly definite. Then there is a complete orthonormal set $\{\phi_k\}$ which is a set of eigenfunctions for each operator and

$$\Gamma\phi_k = \lambda_k\phi_k$$

$$R\phi_k = \mu_k\phi_k \quad .$$

As before, take

$$w_k = \int_{T_1}^{T_2} w(t)\,\phi_k(t)\,dt \quad .$$

Then the log likelihood ratio, formed using the statistics $w_1, \ldots, w_N$, for the hypothesis "signal plus noise present" against the hypothesis "noise alone" is

$$\log \frac{p(w_1, \ldots, w_N;1)}{p(w_1, \ldots, w_N;0)} = -\frac{1}{2}\sum^{N}\left\{\log\left(1+\frac{\lambda_k}{\mu_k}\right) - \frac{\lambda_k}{\mu_k(\lambda_k+\mu_k)}\,w_k^2\right\} \tag{64}$$

Now the condition

$$\sum^{\infty} \frac{\lambda_k^2}{\mu_k^2} < \infty \tag{65}$$

is necessary and sufficient that the detection problem be non-singular (see [11]), in the same sense as the term was used in the preceding section; and if (65) is satisfied the expression for the log likelihood ratio given by Eq. (64) converges with probability one on either hypothesis as $N \to \infty$. We shall, however, assume satisfied the more stringent condition,

$$\sum^{\infty} \frac{\lambda_k}{\mu_k} < \infty \tag{66}$$

which has been called by Hajek the condition for <u>strong</u> equivalence of the signal-plus-noise and noise probability measures [27]. If (66) holds one can write for the limit log likelihood ratio

$$\log \ell(w) = -\frac{1}{2}\sum^{\infty} \log\left(1+\frac{\lambda_k}{\mu_k}\right) + \frac{1}{2}\sum^{\infty} \frac{\lambda_k}{\mu_k(\lambda_k+\mu_k)}\,w_k^2 \tag{67}$$

The first series in Eq. (67) converges, and the second series converges both in mean-square and with probability one on either hypothesis, as will follow from comments immediately below. Since the first series does not depend on w, the statistical inference is determined entirely by the second series. We can thus define a test functional, or test statistic

$$q(w) \triangleq \sum^{\infty} \frac{\lambda_k}{\mu_k(\lambda_k + \mu_k)} w_k^2 \tag{68}$$

for use in maximum-likelihood procedures. Since the random variables w_k are mutually independent under either hypothesis, the w_k^2 are also. The means and variances of q(t) are readily calculated to be:

$$E_0 q(w) = \sum^{\infty} \frac{\lambda_k}{\lambda_k + \mu_k}$$

$$E_1 q(w) = \sum^{\infty} \frac{\lambda_k}{\mu_k}$$

$$\mathrm{var}_0 q(w) = 2 \sum^{\infty} \frac{\lambda_k^2}{(\lambda_k + \mu_k)^2}$$

$$\mathrm{var}_1 q(w) = 2 \sum^{\infty} \frac{\lambda_k^2}{\mu_k^2} \tag{69}$$

All four infinite series converge if condition (66) is satisfied.

A maximum-likelihood test to determine whether signal is present in the received waveform or not consists in comparing q(w) with a predetermined threshhold and answering in the affirmative if q(w) exceeds the threshhold. The probabilities of detection and false detection (false alarm) are governed then by the probability distribution of q(w) and the value of the threshhold. The determination of the distribution of q(w) is complicated and in general the answer does not come in very neat form. We shall not attempt to discuss it here. The problem is a classical one since the distribution of a quadratic form in Gaussian variates has been of interest for a long time for various reasons ([28],[29], [30]).

Under certain conditions the quadratic functional $q(w)$ can be written as an integral

$$q(w) = \int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} Q(t, s)\, w(t)\, w(s)\, dt\, ds \tag{70}$$

In fact, formally, if one puts

$$Q(t, s) = \sum^{\infty} \frac{\lambda_k \phi_k(t) \phi_k(s)}{\mu_k(\lambda_k + \mu_k)} \tag{71}$$

then the right side of Eq. (70) reduces to the series in Eq. (68), and Q satisfies the integral equation

$$\int_{\tau_1}^{\tau_2} \int_{\tau_1}^{\tau_2} R(s, s')[R(t, t') + \Gamma(t, t')]\, Q(t', s')\, dt'\, ds' = \Gamma(t, s), \quad \tau_1 \leq t,\ s,\ \leq \tau_2 \tag{72}$$

as may be verified immediately. The integral equation actually does have the solution $Q(t, s)$ in the space of functions of integrable square on $[\tau_1, \tau_2] \times [\tau_1, \tau_2]$ if

$$\sum \frac{\lambda_k^2}{\mu_k^2(\lambda_k + \mu_k)^2} < \infty. \tag{73}$$

It follows fairly easily that the convergence of the series in (73) implies strong equivalence, i.e., the condition (66). The converse is not true.

It is interesting to note that the conditions (65) and (73) play corresponding roles respectively for this stochastic-signal-in-noise case as (7) and (35) do for the sure-signal-in-noise case. Condition (66), strong equivalence, also implies a kind of stability of the functional $g(f)$ with respect to the underlying probability measures in the commutative case being discussed [17]*.

The relationships just stated precisely can be formally extended to cover the case of white noise; this gives the usual so-called optimum detector for

*The reference cited does not discuss the special commutative case explicitly. The condition given there reduces to (   ) in this case.

detecting Gaussian signals in white Gaussian noise. One has,

$$R(t, u) = N_0 \delta(t - u)$$

and

$$R\phi_k = N_0 \phi_k$$

so that

$$q(w) = \frac{1}{N_0} \sum^{\infty} \frac{\lambda_k}{\lambda_k + N_0} w_k^2 \quad . \tag{74}$$

The first and second moments of q(w) are

$$E_0 q(w) = \sum^{\infty} \frac{\lambda_k}{\lambda_k + N_0}$$

$$E_1 q(w) = \frac{1}{N_0} \sum^{\infty} \lambda_k$$

$$\mathrm{var}_0\, q(w) = 2 \sum^{\infty} \frac{\lambda_k^2}{(\lambda_k + N_0)^2}$$

$$\mathrm{var}_1\, q(w) = \frac{2}{N_0^2} \sum^{\infty} \lambda_k^2 \tag{75}$$

all of which converge since

$$\sum^{\infty} \lambda_k = \int_{T_1}^{T_2} R(t, t)\, dt$$

must converge. It follows that the series in Eq. (74) for q(w) converges in mean-square and with probability one on either hypothesis, so the test functional q(w) is meaningful. Furthermore, the integral equation for Q(t, s), Eq. (72), reduces to

$$N_0 \int_{T_1}^{T_2} \Gamma(t, t')\, Q(t', s)\, dt' = \Gamma(t, s) - N_0^2\, Q(t, s)$$

which has the solution

$$Q(t, s) = \frac{1}{N_0} \sum \frac{\lambda_k}{\lambda_k + N_0} \phi_k(t)\, \phi_k(s) \quad .$$

q(w) is again given by the integral quadratic form of Eq. (71).

In the special case just considered, the commutativity of the operators R and Γ allows the Karhunen-Loeve expansion of either one to reduce the problem to diagonal form. In the general case, a "diagonal form" can be found, but it is much harder to come by. The analysis here does parallel one way of handling the general case, however, (see [ ]).

BIBLIOGRAPHY

1. Davenport, W. B., and Root, W. L., "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill, New York, 1958.
2. Lehman, E. L., "Testing Statistical Hypotheses," John Wiley and Sons, New York, 1959.
3. Fraser, D. A. S., "Nonparametric Methods in Statistics," John Wiley and Sons, New York, 1957.
4. Doob, J. L., "Stochastic Processes," John Wiley and Sons, New York, 1953.
5. Riesz, F. and Nagy, B. Sz., "Functional Analysis," Ungar, 1955.
6. Grenander, U., Stochastic Processes and Statistical Inference, Ark. fur Matematik 1 (17): 195-277, 1950.
7. Parzen, E., An Approach to Time Series Analysis, Ann. Math. Stat., 32(4), 951-989, 1961.
8. Loeve, M., "Probability Theory," D. Van Nostrand, Princeton, 1955.
9. Hajek, J., On a Property of Normal Distribution of any Stochastic Process, Selected Translations in Math. Stat. Prob., 1, 245-253 (original paper, 1958).
10. Feldman, J., Equivalence and Perpendicularity of Gaussian Processes, Pacific J. Math. 8(4), 699-708, 1958.
11. Root, W. L., Singular Gaussian Measures in Detection Theory, Proc. of Symp. on Time Series Analysis, ed. M. Rosenblatt, 292-315, John Wiley and Sons, 1962.
12. Yaglom, A. M., On the Equivalence or Perpendicularity of Two Gaussian Probability Measures in Function Space, Proc. of Symp. on Time Series Analysis, ed. M. Rosenblatt, 327-348, John Wiley and Sons, 1962.
13. Parzen, E., Probability Density Functionals and Reproducing Kernel Hilbert Spaces, Proc. of Symp. on Time Series Analysis, ed. M. Rosenblatt, 155-169, John Wiley and Sons, 1962.
14. Titchmarsh, E. C., "Introduction to the Theory of Fourier Integrals," Oxford, 2nd Ed., 1948.
15. Kelly, E. J. Reed, I. S. and Root, W. L., The Detection of Radar Echoes in Noise, I. J. Soc. Indust. Appl. Math. 8, 309-341, 1960.
16. Dwork, B., Detection of a Pulse Superimposed on Fluctuation Noise, Proc. IRE, 38, 1950.
17. Root, W. L., Stability in Signal Detection Problems, Proc. Symp. Appl. Math. 16, 247-263, 1964.

18. Kaczmarz, S. and Steinhaus, H., "Theorie der Orthogonalreihen," Chelsea, New York, 1951.

19. Zaanen, A.C., "Linear Analysis," North Holland, Amsterdam, 1956.

20. Kadota, T.T., Generalized Maximum Likelihood Test and Minimum Error Probability, To appear in IEEE Trans. Info. Theory.

21. Slepian, D., Some Comments on the Detection of Gaussian Signals in Gaussian Noise, IRE Trans. Info. Theory, 4(2), 65-69, 1958.

22. Feldman, J., Some Classes of Equivalent Gaussian Processes on an Interval, Pacific J. Math., 10(4), 1211-1220, 1960.

23. Capon, J., Radon-Nikodym Derivatives of Stationary Gaussian Measures, Ann. Math. Stat., 35, 517-531, 1964.

24. Kadota, T.T., Optimum Reception of Binary Gaussian Signals, Bell Syst. Tech. Jour.

25. Pitcher, T.S., An Integral Expression for the Log Likelihood Ratio of Two Gaussian Processes, To appear in J. Soc. Indust. Appl. Math.

26. Balakrishnan, A.V., On a Class of Non Linear Estimation Problems, IEEE Trans. on Info. Theory IT-10(4), 314-321, 1964.

27. Hajek, J., Linear Statistical Problems in Stochastic Processes, Czech. Mat. Journal, 87, 404-444, 1962.

28. Robbins, H.E., Distribution of a Definite Quadratic Form, Ann. Math. Stat., 19, 266-270, 1948.

29. Gurland, J., Distribution of Definite and Indefinite Quadratic Forms, Ann. Math. Stat., 26, 122-127, 1955.

30. Slepian, D., Fluctuations of Random Noise Power, Bell Syst. Tech. Jour., 37(1), 163-184, 1958.

APPENDIX

Throughout we have made the assumption that the operator R on $L_2[a, b]$, a and b finite, defined by

$$[Rx](t) = \int_a^b R(t, u)\, x(u)\, du\,, \qquad a \leq t \leq b \tag{A1}$$

where R(t, u) is an autocorrelation function, is strictly positive definite, i.e., that

$$(Rx, x) > 0 \tag{A2}$$

for any non-zero $x \in L_2[a, b]$. Since R is a completely continuous, self-adjoint operator, (A2) implies that it has a complete orthonormal set of eigenfunctions whose corresponding eigenvalues are real and strictly positive.

In some places this assumption is merely a convenience, however for the asymptotic results for stationary noise and long observation times it is essential. We shall therefore prove a sufficient condition that seems adequate to take care of most cases.

Let R(t), $-\infty < t < \infty$, be a continuous autocorrelation function (i.e., a continuous, hermitian symmetric, non-negative definite function). Let x(t) be of integrable square on [a, b] and

$$\int_a^b |x(t)|^2\, dt > 0\ .$$

Extend x(t) for all t, $-\infty < t < \infty$, by taking it to be indentically zero outside [a, b]. Then

$$\begin{aligned}(Rx, x) &= \int_a^b \int_a^b R(t - u)\, x(u)\, \overline{x(t)}\, du\, dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(t - u)\, x(u)\, \overline{x(t)}\, du\, dt\end{aligned}$$

By Bochner's theorem,

$$(Rx, x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i2\pi f(t-u)}\, dG(f)\; x(u)\, \overline{x(t)}\, du\, dt \tag{A3}$$

where G(f) is a bounded monotone non-decreasing function. Since x(t) is zero outside a finite interval, the integral in (A3) is absolutely convergent, so we may write

$$(Rx, x) = \int_{-\infty}^{\infty} dG(f) \int_{-\infty}^{\infty} e^{i2\pi ft} x(t)\, dt \int_{-\infty}^{\infty} e^{-i2\pi fu} x(u)\, du$$

$$= \int_{-\infty}^{\infty} |X(f)|^2\, dG(f) \tag{A4}$$

where X(f) is the Fourier transform of x(t). The function X(f) is analytic, not identically zero, and cannot vanish on any set of points converging to a finite limit. Hence if the points of increase of G(f) contain a convergent set, $(Rx, x) > 0$ by (A4). This is a very weak condition; for example, it is sufficient that G have any continuous part at all. A fortiori it is sufficient that G have a non-zero density function.